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THE APPLICATION OF THE SPIN COEFFICIENT METHOD FOR THE SPACE-LIKE SYMMETRIC ELECTROVAC PROBLEM

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ABSTRACT

To find the solutions of the Einstein-equations of the general relativity theory is a difficult problem. In the stationary case the application of the method of the 3 dimensional relativity and spin coefficients enable one to find new solutions. The problem with space-like symmetry has been investigated less than the stationary case. The method of the 3 dimensional relativity and spin coefficients can be applied for space-like symmetric case too. This paper contains the equivalents of the field equations of the space-like symmetric electrovac problem in 3 dimensional relativity expressed by spin coefficients.

KIVONAT

Az általános relativitáselmélet Einstein-egyenleteinek megoldásait általános esetben nehéz megkeresni. Stacionaritás esetén a 3 dimenziós relativitáselmélet, valamint az ehhez adaptált spinkoefficiens-módszer alkalmazása uj megoldások megtalálását tette lehetővé. Ezek a módszerek abban a jóval kevésbé tanulmányozott esetben is alkalmazhatók, amikor stacionaritás helyett egy térbeli szimmetria áll fenn. Ezen munka a térbeli szimmetriával rendelkező elektrovákuum Einstein- és Maxwell-egyenleteit tartalmazza a 3 dimenziós relativitáselmélet formalizmusával és spinkoefficiensek segitségével felirva.

РЕЗЮМЕ

Нахождение решений уравнений Эйнштейна общей теории относительности в общих случаях является трудной задачей. В случае стационарности применение трехмерной теории относительности, а также адаптированного для такого случая метода спиновых коэффициентов представило возможность нахождения новых решений. Эти методы применяются и в случаях пространственноподобной симметрии вместо стационарности. Настоящая работа содержит уравнения Эйнштейна и Максвелла пространственно-подобного симметричного электровакуума, в случае трехмерной теории относительности, содержащие коэффициенты спина.

1. INTRODUCTION

In stationary case the application of the 3 dimensional relativity and the spin coefficients [1], [2], [3] enables one to find new vacuum and electrovac solutions [4], [5], [6].The space-like symmetric cases are investigated less than the stationary cases. The method of the 3 dimensional relativity and the spin coefficients is applicable in space-like symmetric cases too [7]. In this paper we shall investigate the electrovac problem with one space-like symmetry.This paper contains the equivalents of the Einstein- and Maxwell-equations in the 3 dimensional relativity expressing these by spin coefficients.

The energy-momentum tensor of the electrovac problem has the following form:

$$T_{\mu\nu} = F_{\mu\rho}\, F^{\rho}_{\nu} - \frac{1}{4}\, g_{\mu\nu}\, F_{\rho\sigma}\, F^{\rho\sigma} \quad . \qquad /1.1/$$

The Einstein- and Maxwell-equations are:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \kappa T_{\mu\nu} \quad ; \qquad /1.2/$$

$$F^{\mu\nu}{}_{;\nu} = 0 \quad ;$$

and we have:

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \quad . \qquad /1.3/$$

Here A^{μ} is the electromagnetic vector potential. The field equations /1.2/ are invariant against the following transformation /duality rotation/:

$$F'_{\mu\nu} = \cos\alpha\, F_{\mu\nu} + \sin\alpha \cdot \epsilon_{\mu\nu\rho\sigma}\, F^{\rho\sigma}\, \sqrt{-g} \qquad /1.4/$$

α is a real constant.

Now we shall show that the space-like symmetric electrovac problem can be investigated analogously to the stationary problem.

2. SPACE-LIKE SYMMETRIC ELECTROVAC PROBLEM

The electrovac field has a space-like symmetry if the Killing-equation

$$K_{\mu;\nu} + K_{\nu;\mu} = 0 \quad ; \qquad \mu = 0,1,2,3 \qquad /2.1/$$

has a space-like K^{μ} solution and the Lie-derivative of the elecromagnetic vector potential vanishes along this K^{μ} field:

$$A_{\mu;\rho} K^{\rho} - A^{\rho} K_{\mu;\rho} = 0 \quad . \qquad /2.2/$$

The coordinate system can be chosen such that

$$K^{\mu} = \delta^{\mu}_{3} \quad . \qquad /2.3/$$

The conditions /2.3/ are preserved by the following transformations:

$$\begin{aligned} z' &= z + F(x^{i}) \\ x^{i'} &= x^{i'}(x^{k}) \\ z &\equiv x^{3} \\ i &= 0,1,2 \ . \end{aligned} \qquad /2.4/$$

We shall confine ourselves to these transformations.

Let us write the line element of the 4 dimensional spacetime in the form:

$$\begin{aligned} d\tilde{s}^2 &= -f^{-1} ds^2 + f\left(dz + \omega_i \, dx^i\right)^2 \quad ; \\ ds^2 &\equiv g_{ik} \, dx^i \, dx^k \quad . \end{aligned} \qquad /2.5/$$

/The tilde denotes the 4 dimensional quantities./ This form is general because $f = K_{\mu} K^{\mu} < 0$.

It is suitable to introduce the following quantities /similarly to the stationary case [6]/:

$$\varepsilon \equiv f - |\phi|^2 + i\varphi \; ; \qquad B_{,i} \equiv e_{ik\ell}\left(A^{k|\ell} - \omega^k A_3^{1\ell}\right) g\sqrt{g} \quad ;$$

$$\phi \equiv \sqrt{-\frac{K}{2}}\left(A_3 + iB\right) \; ; \quad \varphi_{,i} \equiv e_{ik\ell}\, \omega^{k|\ell} g^2\sqrt{g} + 2\mathrm{Im}\left(\phi\bar{\phi}_{,i}\right) . \qquad /2.6/$$

/The stroke signifies the 3 dimensional covariant derivation./ Now the new form of the transformation /1.4/ is the following:

$$\phi' = e^{i\alpha}\,\phi \; . \qquad /2.7/$$

For ε and ϕ we get the same equations as in the stationary case [3]:

$$\left(\mathrm{Re}\varepsilon + |\phi|^2\right)\Delta\varepsilon = \left(\nabla\varepsilon + 2\bar{\phi}\nabla\phi\right)\nabla\varepsilon$$

$$\left(\mathrm{Re}\varepsilon + |\phi|^2\right)\Delta\phi = \left(\nabla\varepsilon + 2\bar{\phi}\nabla\phi\right)\nabla\phi$$

$$R_{ik} = -\frac{1}{2\left(\mathrm{Re}\varepsilon + |\phi|^2\right)^2}\,\mathrm{Re}\Big\{\varepsilon_{,i}\,\bar{\varepsilon}_{,k} + 4|\phi|^2\,\phi_{,i}\,\bar{\phi}_{,k} + \qquad /2.8/$$

$$+ 2\phi\bar{\phi}_{,i}\varepsilon_{,k} + 2\phi\bar{\phi}_{,k}\varepsilon_{,i} - 4(\mathrm{Re}\varepsilon)\,\phi_{,i}\,\bar{\phi}_{,k}\Big\}$$

We introduce the quantities ξ, q similarly to the stationary case [3]. /Their equations are identical with the equations of the stationary case./ The definitions of these quantities are:

$$\varepsilon \equiv \frac{\xi - 1}{\xi + 1} \; , \qquad \phi \equiv \frac{q}{\xi + 1} \qquad /2.9/$$

If we confine ourselves to the case $q = q^o =$ constant, we get again:

$$\left(\xi\bar{\xi} + q^o\bar{q}^o - 1\right)\Delta\xi = 2\bar{\xi}(\nabla\xi)^2 \qquad /2.10/$$

$$R_{ik} = -2\left(\xi\bar{\xi} + q^o\bar{q}^o - 1\right)^{-2}\left(1 - q^o\bar{q}^o\right)\mathrm{Re}\left(\xi_{,i}\,\bar{\xi}_{,k}\right) .$$

The quantity ξ' defined by the following way:

$$\xi' \equiv \frac{\xi}{\sqrt{1 - q^o\bar{q}^o}} \; ; \quad q^o\bar{q}^o < 1 \qquad /2.11/$$

is a solution of the vacuum equations. This correspondance between the vacuum problem and the electrovac one with $q = q^o$, $|q^o| < 1$ is suitable for construction of the electrovac counterparts of the corresponding vacuum solutions, similarly to the stationary case.

In the space-like symmetric case $\mathrm{Re}\varepsilon + |\phi|^2 < 0$. Thus we introduce the following 3-vectors:

$$\underline{G} \equiv \frac{\nabla\varepsilon + 2\bar{\phi}\nabla\phi}{2\left(\mathrm{Re}\varepsilon + |\phi|^2\right)} \quad ; \quad \underline{H} \equiv \frac{\nabla\phi}{\sqrt{-\left(\mathrm{Re}\varepsilon + |\phi|^2\right)}} \qquad /2.12/$$

Now the new forms of the eqs. /2.8/ are:

$$\begin{aligned}
(\nabla - \underline{G})\underline{G} &= -\left(\underline{G}\underline{\bar{G}} + \underline{H}\underline{\bar{H}}\right)\\
\nabla\times\underline{G} &= \underline{G}\times\underline{\bar{G}} + \underline{H}\times\underline{\bar{H}}\\
(\nabla-\underline{G})\underline{H} &= \tfrac{1}{2}(\underline{G}-\underline{\bar{G}})\underline{H}\\
\nabla\times\underline{H} &= -\tfrac{1}{2}(\underline{G}+\underline{\bar{G}})\times\underline{H}\\
R_{ik} &= -\left(G_i\bar{G}_k + G_k\bar{G}_i + H_i\bar{H}_k + H_k\bar{H}_i\right)\\
(\underline{A}\times\underline{B})_i &\equiv \epsilon_{ik\ell}\, A^k\, B^\ell \sqrt{g} \ .
\end{aligned} \qquad /2.13/$$

Thus there does not exist solution with flat background space except the empty Minkowskian space.

3. THE NEW FORMS OF THE FIELD EQUATIONS

We can write down the equivalents of the eqs. /2.13/ by means of spin coefficients. The details of this method are given in [2], [6] [7]. We define a basic vector triad and complex rotational coefficients in the following way:

$$z^i_{\underline{p}} \equiv \left(\ell^i, m^i, \bar{m}^i\right) ; \qquad V_{\underline{p}} \equiv V_i\, z^i_{\underline{p}}$$

$$g_{\underline{mn}} \equiv z_{\underline{m}}^{\ r}\, z_{\underline{n}r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D \equiv \ell^i \frac{\partial}{\partial x^i} \qquad /3.1/$$

$$\underline{p} = 0,+,- . \qquad \delta \equiv m^i \frac{\partial}{\partial x^i}$$

$$\rho \equiv m_{i|k}\, \ell^i \bar{m}^k \quad ; \quad \sigma \equiv m_{i|k}\, \ell^i m^k \quad ;$$
$$\kappa \equiv m_{i|k}\, \ell^i \ell^k \quad ; \quad \tau \equiv m_{i|k}\, \bar{m}^i \bar{m}^k \quad ; \qquad /3.2/$$

$$\epsilon \equiv m_{i\,k}\, \bar{m}^i \ell^k \quad .$$

The commutators of the differential operators /3.1/ are:

$$D\delta - \delta D + (\bar{\rho}+\epsilon)\delta + \sigma\bar{\delta} - \kappa D = 0 \quad .$$
$$/3.3/$$
$$\delta\bar{\delta} - \bar{\delta}\delta - \tau\delta + \bar{\tau}\bar{\delta} + (\rho-\bar{\rho})D = 0 \quad .$$

Instead of the eqs. /2.13/ we get the following equations:

$$D\rho - \bar{\delta}\kappa = \kappa\tau + \kappa\bar{\kappa} - \rho^2 - \sigma\bar{\sigma} - G_o\bar{G}_o - H_o\bar{H}_o$$

$$D\sigma - \delta\kappa = -(\rho+\bar{\rho}+2\epsilon)\sigma - \kappa\bar{\tau} + \kappa^2 + 2G_+\bar{G}_+ + 2H_+\bar{H}_+$$

$$D\tau - \bar{\delta}\epsilon = -\kappa\bar{\sigma} + \bar{\kappa}\rho + \epsilon\tau + \bar{\kappa}\epsilon - \rho\tau + \bar{\sigma}\bar{\tau} - \bar{G}_oG_- - G_o\bar{G}_- - \bar{H}_oH_- - H_o\bar{H}_-$$

$$\delta\rho - \bar{\delta}\sigma = 2\sigma\tau - (\rho-\bar{\rho})\kappa - G_o\bar{G}_+ - \bar{G}_oG_+ - H_o\bar{H}_+ - \bar{H}_oH_+$$

$$\delta\tau + \bar{\delta}\bar{\tau} = -2\tau\bar{\tau} - \sigma\bar{\sigma} + \rho\bar{\rho} - \epsilon(\rho-\bar{\rho}) - G_o\bar{G}_o - G_-\bar{G}_+ - G_+\bar{G}_- - H_o\bar{H}_o - H_-H_+ - \bar{H}_-H_+$$

$$DG_o - \bar{\delta}G_+ - \delta G_- = (-\rho-\bar{\rho}+G_o-\bar{G}_o)G_o + (\bar{\kappa}+\tau-G_-+\bar{G}_-)G_+ + (\kappa+\bar{\tau}-G_++\bar{G}_+)G_- - H_o\bar{H}_o+H_-\bar{H}_-+\bar{H}_-H_+$$

$$DH_o-\bar{\delta}H_+-\delta H_- = \left(-\rho-\bar{\rho}+\tfrac{3}{2}G_o - \tfrac{1}{2}\bar{G}_o\right)H_o + \left(\bar{\kappa}+\tau-\tfrac{3}{2}G_- + \tfrac{1}{2}\bar{G}_-\right)H_+ + \left(\kappa+\bar{\tau} -\tfrac{3}{2}G_+ + \tfrac{1}{2}\bar{G}_+\right)H_-$$

$$\delta G_o - DG_+ = \sigma G_- + (\bar{\rho}+\bar{G}_o+\epsilon)G_+ - (\kappa+\bar{G}_+)G_o + \bar{H}_oH_+ - \bar{H}_+H_o$$

$$\delta H_o - DH_+ = \sigma H_- + \left[\bar{\rho} + \tfrac{1}{2}(G_o+\bar{G}_o) + \epsilon\right]H_+ - \left[\kappa + \tfrac{1}{2}(G_+ + \bar{G}_+)H_o\right]$$

$$\bar{\delta}G_o - DG_- = (\rho-\epsilon+\bar{G}_o)G_- + \bar{\sigma}G_+ - (\bar{\kappa}+\bar{G}_-)G_o + H_oH_- - H_o\bar{H}_-$$

$$\bar{\delta}H_o - DH_- = \left[\rho-\epsilon + \tfrac{1}{2}(G_o+\bar{G}_o)\right]H_- + \bar{\sigma}H_+ - \left[\bar{\kappa} + \tfrac{1}{2}(G_- + \bar{G}_-)\right]H_o$$

$$\bar{\delta}G_+ - \delta G_- = (\bar{\tau}+\bar{G}_+)G_- - (\tau+\bar{G}_-)G_+ + (\rho-\bar{\rho})G_o + \bar{H}_+H_- - \bar{H}_-H_+$$

$$\bar{\delta}H_+ - \delta H_- = \left[\bar{\tau} + \tfrac{1}{2}(G_+ + \bar{G}_+)\right]H_- - \left[\tau + \tfrac{1}{2}(G_- + \bar{G}_-)\right]H_+ + (\rho-\bar{\rho})H_o \quad .$$

$$/3.4/$$

4. THE EIGENRAYS

In many cases the eqs. /3.4/ can be reduced by the suitable choice of the z^i_f triad. Let be a complex 3-vector field:

$$\underline{u} = \underline{a} + i\underline{b} \quad . \qquad /4.1/$$

If for an 3-vector ℓ the following algebric requirement

$$\underline{a} - (\underline{a}\underline{\ell})\underline{\ell} + \underline{b}\times\underline{\ell} = 0 \qquad /4.2/$$

is fulfilled, $\underline{\ell}$ is the tangent to the eigenrays of this field [2].

The $\underline{z}_o$ vector of the $\underline{z}_f$ triad can be chosen as the tangent of eigenrays except the following cases:

$$\begin{aligned} &1. \quad a = 0 \quad \text{and} \quad \underline{b}\underline{b} \leq 0 \; . \\ \text{or} \quad &2. \quad \underline{a} \neq 0, \quad \underline{b} = 0 \quad \text{and} \quad \underline{a}\underline{a} \leq 0 \; . \qquad /4.3/ \\ \text{or} \quad &3. \quad \underline{a} \neq 0, \quad \underline{b} \neq 0 \quad \text{and} \quad \underline{a}\underline{a} = \underline{b}\underline{b} = \underline{a}\underline{b} = 0 \; . \end{aligned}$$

When we investigate the eigenrays of the gravitational /$\underline{G}$/ field, we have

$$\begin{aligned} \underline{a} &= \nabla\left(f - |\phi|^2\right) \\ \underline{b} &= \nabla\varphi \; . \end{aligned} \qquad /4.4/$$

If on the other side we investigate the eigenrays of the electromagnetic /$\underline{H}$/ field then

$$\begin{aligned} \underline{a} &= \nabla \mathrm{Re}\phi \\ \underline{b} &= \nabla \mathrm{Im}\phi \; . \end{aligned} \qquad /4.5/$$

If the conditions /4.3/ are not fulfilled for one of these fields, we can choose the $\underline{\ell}$ vector of the triad tangent to the eigenrays of this field. In this case

$$G_- = 0 \quad \text{or} \quad H_- = 0 \; . \qquad /4.6/$$

Finally we can make $\epsilon = 0$ by a rotation of triad about the axis $\underline{\ell}$ /[2], [7]/ and the freedom

$$\underline{\ell}' = \ell \; ; \qquad \underline{m}' = e^{iC^o}\,\underline{m} \; ; \qquad C^o \text{ is real, and } DC^o = 0 \,. \qquad /4.7/$$

remains. If $D\chi^o = 0$, we can make $\mathrm{Im}\chi^o = 0$ by the transformation /4.7/.

5. CONCLUSION

These equations are similar to the equations of the stationary electrovac problem and can be solved in the same way. Further investigations on this will appear later.

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